DECAY OF A VERTICAL TORNADO-LIKE VORTEX

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The flow in tornado-like and hollow vortices was studied in Nikulin [1]. In the long-wave approximation, equations were obtained which are analogous to the equations of shallow water vortices [2]. For steady vertical tornado-like vortices, whose core fluid is lighter than the surrounding fluid, a sharp criterion was established dividing the cases where the solution is continued to a finite or infinite height.

In this paper, as a supplement to Nikulin [1], the case is studied where the fluid in the core of the vortex is heavier than the surrounding fluid, and the direction of the force of gravity coincides with that of the vertical velocity. In addition, an analytic example is constructed in which the solution, bounded in height, being investigated in Nikulin [1] for a vortex with a light core is continuously distributed over the entire half-space.

1. Statement of Problem. We examine a half-space filled with an inviscid incompressible fluid in a gravity field. The flow is assumed to be steady and rotationally symmetric. We introduce a cylindrical system of coordinates ($r\theta z$) (r is the radius, θ the azimuthal angle, and z is the axis of symmetry, directed opposite the force of gravity). The boundary which separates the core of the vortex from the outer flow is denoted by $r = r_0(z)$; the outer flow is located in the region $r > r_0(z)$. The density of the fluid in the outer flow is assumed to be constant. At the core boundary, a jump in the density and in the component of the velocity tangential to the core is possible. Length, velocity, and density scales are introduced to transform to dimensionless quantities. For the unit length, the characteristic scale of change along the z axis is adopted; for unit velocity, the rotational component of the velocity at $r = r_0$, $z = 0$; and for unit density, the density of the outer flow. In this case, the characteristic pressure and acceleration will be equal to unity. We denote by δ the dimensionless r_0 at $z = 0$. Below, all quantities are given in dimensionless form unless it is specifically stated otherwise.

The velocity components corresponding to ($r\theta z$) are written as (uvw); p is the pressure; ρ the density; g the acceleration of gravity. The outer flow is assumed to be known and is prescribed in a form satisfying the equations of motion:

$$u = w = 0,\ v = \delta/r,\ p = -\delta^2/(2r^2) - gz. \tag{1.1}$$

The flow in the core is studied in the long-wave approximation along the z axis. We carry out a scaling of the coordinates and functions:

$$r^2 \to \delta^2\eta,\ z \to z,\ 2ur \to \delta^2 q,\ vr \to \delta A,$$
$$w \to w,\ \rho \to \rho,\ p \to p,\ g \to g.$$

The boundary $r_0(z)$ goes over to $\eta_0(z)$. After substitution, the equations of motion and continuity take on the form

$$\begin{gathered} q\frac{\partial A}{\partial \eta} + w\frac{\partial A}{\partial z} = 0, \\ \frac{\rho\delta^2}{2}\left(q\frac{\partial q}{\partial \eta} - \frac{q^2}{2\eta} + w\frac{\partial q}{\partial z}\right) - \frac{\rho A^2}{\eta} = -2\eta\frac{\partial p}{\partial \eta}, \\ \rho\left(q\frac{\partial w}{\partial \eta} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} - \rho g, \\ \frac{\partial q}{\partial \eta} + \frac{\partial w}{\partial z} = 0,\quad q\frac{\partial \rho}{\partial \eta} + w\frac{\partial \rho}{\partial z} = 0. \end{gathered} \tag{1.2}$$

The boundary conditions at the axis of symmetry and the boundary of the core become:

$$q = A = 0, \qquad \eta = 0; \tag{1.3}$$

$$p = -1/(2\eta_0) - gz, \qquad \eta = \eta_0; \tag{1.4}$$

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$$q = w(\partial \eta_0/\partial z), \qquad \eta = \eta_0. \tag{1.5}$$

Condition (1.4) follows from (1.1) and the requirement of continuity of pressure at the core boundary. Equation (1.5) is a kinematic condition.

It is assumed that $\delta \ll 1$. The terms in (1.2) which are proportional to δ^2 are neglected. The resultant system is transformed as in Nikulin [1]. New independent variables z', ν, $\nu \in [01]$ are introduced in accordance with the relations $z = z'$, $\eta = R(z', \nu)$, where R satisfies

$$w(\partial R/\partial z') = q \tag{1.6}$$

and boundary conditions

$$R(z', 0) = 0, \; R(z', 1) = \eta_0, \tag{1.7}$$

$R(0, \nu)$ is an arbitrary single-valued continuous function. Boundary conditions (1.3) (for q) and (1.5) are automatically satisfied for this definition of R. The unknown boundary $\eta_0(z)$ is changed into the known boundary $\nu = 1$. In the variables z', ν, the system (1.2) takes on the form (henceforth, the prime on z' is omitted)

$$\begin{gathered} w\frac{\partial A}{\partial z} = 0, \quad \frac{\rho A^2}{2R^2}\frac{\partial R}{\partial \nu} = \frac{\partial p}{\partial \nu}, \\ \rho\frac{\partial R}{\partial \nu} w \frac{\partial w}{\partial z} = -\frac{\partial R}{\partial \nu}\frac{\partial p}{\partial z} + \frac{\partial R}{\partial z}\frac{\partial p}{\partial \nu} - \rho g \frac{\partial R}{\partial \nu}, \\ \frac{\partial q}{\partial \nu} + \frac{\partial R}{\partial \nu}\frac{\partial w}{\partial z} - \frac{\partial R}{\partial z}\frac{\partial w}{\partial \nu} = 0, \quad w\frac{\partial \rho}{\partial z} = 0. \end{gathered} \tag{1.8}$$

It follows from the first and last equation that $A = A(\nu)$, $\rho = \rho(\nu)$. The quantity p is eliminated from this, after integrating the second equation in (1.8) and using (1.4). With the help of (1.6), q is eliminated. As a result, we obtain

$$\rho w \frac{\partial w}{\partial z} = -\frac{1 - \rho_1 A_1^2}{2R_1^2}\frac{dR_1}{dz} + \frac{\partial}{\partial z}\int_\nu^1 \frac{1}{2R}\frac{d(\rho A^2)}{d\nu} d\nu + (1 - \rho) g, \quad \frac{\partial}{\partial z}\left(w\frac{\partial R}{\partial \nu}\right) = 0. \tag{1.9}$$

Here, R_1, A_1, ρ_1 are the values of R, A, and ρ at $\nu = 1$ (at the core boundary). System (1.9) is solved with initial data at $z = 0$. We set $w = w_0(\nu)$, $R = \nu$ at $z = 0$.

Nikulin [1] studied the case $A = 0$, $\rho = \text{const} < 1$. In the present work, the results are generalized to $q > 1$.

<u>THEOREM.</u> Let $A = 0$, $\rho = \rho_1 = \text{const}$, $w_0(\nu)$ be bounded and $w_0(\nu) \geqslant \gamma > 0$, γ constant, and

$$\lambda = \frac{1}{2\rho_1}\int_0^1 \frac{d\nu}{w_0^2}.$$

The cases $\rho_1 < 1$ and $\rho_1 > 1$ are considered separately.

1. Let $\rho_1 < 1$. Then, if $\lambda < 1$, the solution exists for all $z > 0$, $\nu \in [01]$, and $R \to 0$, $w \to \infty$ monotonically for $z \to \infty$. If $\lambda > 1$, then the solution exists only for $z \leqslant \ell$,

$$l = \frac{\rho_1}{2(1 - \rho_1) g}\left(\frac{1}{\rho_1} - \gamma^2 - \frac{1}{\rho_1 \int_0^1 \frac{w_0 d\nu}{\sqrt{w_0^2 - \gamma^2}}}\right), \tag{1.10}$$

and $w = 0$, $\partial R/\partial \nu = \infty$ at $z = \ell$ for ν given by the equation $w_0(\nu) = \gamma$.

2. Let $\rho_1 > 1$. In addition, it is assumed that $(w_0 - \gamma)^{-3/2}$ is not integrable on [01]. Then for any $\lambda > 0$, the solution exists only for finite z, $z \leqslant h$. For $z \to h$, the derivative $\partial w/\partial z \to \infty$, $\partial R/\partial z \to -\infty$, if $\lambda > 1$; and $\partial w/\partial z \to -\infty$, $\partial R/\partial z \to \infty$, if $\lambda < 1$. For $\lambda > 1$, w grows monotonically, while R dies out with increasing z. For $\lambda < 1$, w dies out monotonically while R grows with increasing z.

<u>Proof.</u> The proof for $\rho_1 < 1$ is given in Nilkulin [1]. Let us examine the case $\rho_1 > 1$.

In (1.9) we set $A = 0$, $\rho = \rho_1$. The system is integrated from 0 to z. An expression for R is obtained by integrating the second equation over ν. Denoting $\varphi = w^2 - w_0^2$, we obtain

$$\varphi - \frac{1}{\rho_1 R_1} = \frac{2(1 - \rho_1)}{\rho_1} gz - \frac{1}{\rho_1}, \quad R = \int_0^\nu \frac{w_0 d\nu}{(w_0^2 + \varphi)^{1/2}} \tag{1.11}$$

(R_1 is the value of R at $\nu = 1$). From the first equation it follows that $\varphi = \varphi(z)$. From this, by studying the implicit relation $\varphi(z)$, given by (1.11), the dependence of the functions R and w on z can be determined.

The left-hand side of the first equation in (1.11) is denoted by $f(\varphi)$. Then

$$f'(\varphi) = 1 - \frac{1}{2\rho_1 R_1^2} \int_0^1 \frac{w_0 d\nu}{(w_0^2 + \varphi)^{3/2}},$$
$$f''(\varphi) = \frac{3}{4\rho_1 R_1^2} \int_0^1 \frac{w_0 d\nu}{(w_0^2 + \varphi)^{5/2}} - \frac{1}{2\rho_1 R_1^3} \left(\int_0^1 \frac{w_0 d\nu}{(w_0^2 + \varphi)^{3/2}} \right)^2. \tag{1.12}$$

The second term in $f''(\varphi)$ is estimated with the help of the Bunyakovskii inequality [1]. It is found that $f''(\varphi) > 0$ for finite $\varphi \geqslant -\gamma^2$. Whence it follows that $f'(\varphi)$ is a monotonically growing function. From (1.12) and the additional assumption for $\rho_1 > 1$, it follows that $f'(\varphi) \to -\infty$ for $\varphi \to -\gamma^2$ and $f'(\varphi) \to 1$ for $\varphi \to \infty$. Thus, $f'(\varphi)$ has a single zero in the interval $(-\gamma^2, \infty)$, which is attained when $\varphi = \varphi_*$, $-\gamma^2 < \varphi_* < \infty$. It is obvious that the function $f(\varphi)$ has an absolute minimum at the point φ_*; φ_* is determined from the solution to the integral equation $f'(\varphi_*) = 0$.

From the definition of f comes

$$\frac{df}{dz} = \frac{df}{d\varphi}\frac{d\varphi}{dz} = -\frac{2(\rho_1 - 1)}{\rho_1} g < 0. \tag{1.13}$$

From this it follows that the sign of $d\varphi/dz$ is given by the sign of $f'(\varphi)$, and in view of the monotonicity of $f'(\varphi)$, by the sign of $f'(0)$. Since $\varphi(0) = 0$, then $f'(0) = 1 - \lambda$. Let $\lambda < 1$. Then $f'(0) > 0$ and φ dies off monotonically with growth in z from 0 to φ_*. In this case, because $f'(0) > 0$ and the monotonicity of $f'(\varphi)$ $\varphi_* < 0$. For $\varphi \to \varphi_*$, $z \to h$, where h is computed from (1.13) with $\varphi = \varphi_*$. It is easy to see that h will be finite. For $z > h$, the solution does not exist, since the right-hand side of the first equation in (1.11) dies off with growing z, while the left-hand side has an absolute minimum at $z = h\ (\varphi = \varphi_*)$. It follows from (1.13) that $d\varphi/dz \to -\infty$ for $z \to h$ (since in this case $\varphi \to \varphi_*$). Then from (1.11) and the definition of φ it follows that $\partial w/\partial z \to -\infty$, $\partial R/\partial z \to \infty$ for $z \to h$, the value of w dies off while R grows with increasing z from 0 to h.

Let $\lambda > 1$. Reasoning as before, it can be shown that the solution also exists only for bounded $z \leqslant h$, for which $\varphi_* > 0$, $d\varphi/dz \to \infty$, $\partial w/\partial z \to \infty$, $\partial R/\partial z \to -\infty$ for $z \to h$, w increases, while R dies off with increasing z from 0 to h. The theorem is proved.

The obtained results are generalized when the direction of g and w coincide. The formulation of the theorem is unchanged, except that the cases $\rho_1 < 1$ and $\rho_1 > 1$ exchange places.

2. Model of the Decay of a Vortex. An analytical example of a continuous continuation of the solution over the entire upper half-space is constructed for a flow whose parameters satisfy the conditions of the theorem and the inequalities $\rho_1 < 1$, $\lambda > 1$. In addition, it is assumed that $w_0(\nu)$ has a single minimum equal to γ at the point $\nu = 0$, and the function $(w_0 - \gamma)^{-1/2}$ is integrable at zero ($\nu = 0$).

According to the theorem, the solution will exist only for $z \leqslant \ell$ [ℓ is determined from (1.10)], in which case the vertical velocity $w = 0$ while $\partial R/\partial\nu = \infty$ at the point $z = \ell$, $\nu = 0$. From this it is possible to divide the continuation of the solution for the level $z = \ell$ by the line $R = 0$ at the point $z = \ell$, $\nu = 0$ in a plane containing the axis.

Thus, it is assumed that the structure of the flow in a plane containing the axis has the form shown in Fig. 1. At the boundary between regions I and III, p, (uvw), R and ρ are assumed to be continuous. At the boundaries between II and III, between III and the outer flow, the pressure is continuous and the kinematic condition is satisfied. From the continuity of the velocity component, according to (1.6), and the definition of φ, it follows that $\partial R/\partial z$ and φ are continuous at the boundary $z = \ell$.

It is assumed that the region II is filled with a fluid at rest with constant density ρ_2. In region III, the flow will be described by the first equation of system (1.11) and, in place of the second equation, by

$$R = R_1 - \int_\nu^1 \frac{w_0 d\nu}{(w_0^2 + \varphi)^{1/2}}, \tag{2.1}$$

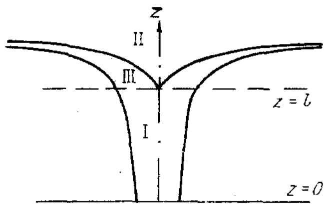

Fig. 1

due to the change in conditions on R at $\nu = 0$. This equation is also valid in I. Because of the condition on ρ at $z = \ell$, the density in region III will be equal to ρ_1. The function $\varphi(z)$ by construction is assumed to be continuous at $z = \ell$. Then from (2.1) it follows that the function $R(\nu, z)$ is continuous in z for all ν. Because of the second equation in (1.8) and the condition A = 0, the pressure in regions II and III is the same for equal z, and according to (1.4), is written as

$$p = -1/2R_1 - gz. \tag{2.2}$$

This ensures the continuity of the pressure at the boundaries. On the other hand, from the equation of hydrostatics, it follows that $p = -\rho_2 gz + \text{const}$ in region II. From this, and from (2.2) we determine R_1:

$$1/R_1 = 1/R_{10} - 2(1 - \rho_2)g(z - l). \tag{2.3}$$

Here R_{10} (the value of R_1 at $z = \ell$) is expressed according to (1.11) with $\varphi = -\gamma^2$ [1]. Substitution of (2.3) into the first equation in (1.11) gives a value of φ in region III, and φ into (2.1) gives the value of R.

The requirement that $\partial R/\partial z$ be continuous at $z = \ell$ imposes limitations on ρ_2. As has been proved, we have from (1.13) that $d\varphi/dz = 0$ in region I for $z = \ell$. From this it follows that $dR/dz = dR_1/dz$ in I for $z = \ell$. The quantity dR_1/dz is determined using (1.11)-(1.13), considering that $\varphi = -\gamma^2$ at $z = \ell$ [1]. Then $dR_1/dz = 2(1 - \rho_1)gR_1^2$. In region III, dR/dz is computed from (2.1) with accout taken of (2.3) and the first equation in (1.11). It is found that dR/dz is the same as in region I if $\rho_2 = \rho_1$.

Thus, the conditions at the region boundaries, and Eqs. (1.11) and (2.1) are satisfied. Region II is filled with a fluid at rest with density ρ_1. The outer boundary R_1 of region III is determined according to (2.3) with $\rho_2 = \rho_1$. The value of φ in III is constant and equal to $-\gamma^2$. The boundary between regions II and III is determined according to (2.1) for $\varphi = -\gamma^2$. From this and from (2.3) it follows that the height in region III is bounded by the quantity

$$\Delta l = 1/[2R_{10}(1 - \rho_1)g].$$

Then, taking (1.10) and the expression for R_{10} into account, we write the total height of the ascent of the fluid as

$$H = l + \Delta l = \frac{\rho_1}{2(1-\rho_1)g}\left(\frac{1}{\rho_1} - \gamma^2\right).$$

Note that H depends only on the minimum value of w_0, equal to γ, but does not depend on the distribution of $w_0(\nu)$. Using the fact that in this case, $\lambda > 1$, it is easy to obtain $\gamma^2 < 1/(2\rho_1)$. From this we have a bound on H:

$$1/[4(1 - \rho_1)g] < H < 1/[2(1 - \rho_1)g].$$

These inequalities make it possible to estimate H to order of magnitude without knowing w_0. For dimensional quantities, this estimate has the form

$$H \approx \rho_0 v_0^2/[2g(\rho_0 - \rho_1)] \tag{2.4}$$

(v_0 is the rotational component of the velocity at the boundary of the core at z = 0, and ρ_0 is the density of the fluid in the outer region).

3. Discussion of Results, and Comparison with Observations. In the case $\rho_1 > 1$, $\lambda > 1$ (in accordance with point 2 of the theorem), the vertical velocity w and the rotational component of the velocity at the core boundary v (since $v = 1\sqrt{R_1}$) grow with increasing z. This perhaps explains the engulfing of fluid by the whirlwind [3], and also is one of the mechanisms of intensifying the rotation of the whirlwind core. Note that for $\rho_1 > 1$, the solution ceases to exist, because the derivatives of the unknown functions tend toward infinity. This

property is analogous to the "gradient catastrophe" in gasdynamics, which leads to the formation of shock waves. A detailed analogy suggests that at the height where the smooth solution ceases to exist, there is a sharp change in the core of the vortex, with a transition to a new solution with other parameters.

The structure of the flow for $\rho_1 < 1$, $\lambda > 1$, shown in Fig. 1 is of interest. From experiments with vortices obtained during heating of an underlying surface [4, 5], and observations of dust devils [6, 7], it is known that the core of the vortex, which is usually made visible by fine particles, sharply drops in visibility at some height and disappears at the point where the structure of the flow changes [4, 5]. The sudden disappearance of the visible core can be explained by the fact that during the spreading out of the fluid in region III, its thickness decreases, and therefore, it rapidly mixes with the surrounding medium and becomes invisible. Such a flow pattern can model the fundamental features of the process of vortex decay.

Using (2.4), it is possible to obtain quantitative estimates and compare these with observations. The height at which a sharp change in the structure of the flow occurs is taken as the height of the vortex in experiments [4, 5]. In the calculations, it is convenient to express the difference in densities by the difference in temperatures. For $\rho_0 - \rho_1 \ll \rho_0$, we have $(\rho_0 - \rho_1)/\rho_0 \approx (T_1 - T_0)/T_0$ (T_1 is the average temperature in the core, and T_0 is the temperature of the surrounding medium). Then

$$H \approx v_0^2 T_0/[2g(T_1 - T_0)]. \tag{3.1}$$

A similar expression for the height of the vortex was obtained in Nikulin [1] for the particular case when w_0 = const and does not depend on ν. In the present work, it has been shown that this estimate is valid for any $w_0(\nu)$ satisfying the condition $\lambda > 1$ when considering the height of ascension of the fluid in region III. Comparison of (3.1) with experiment [4, 5] and observations of dust devils [6-8] was done in Nikulin [1]. It was shown that to an order of magnitude, calculations agree with laboratory measurements and observations of dust devils.

Thus within the proposed theoretical model, the flow in the core of a steady tornado-like vortex has been studied, with account taken of the variability of vertical component of the velocity in a horizontal cross section of the core. As a result of this, the possibility has been established of continuing the continuous solution to infinity or to some bounded height, for which the radius of the core remains finite. The solutions which cannot be continued have been classified: the solution ceases to exist either due to the vanishing of the vertical velocity, or due to the increase without bounds of the derivatives. In the first case, it is possible to draw an analogy with the flow in a boundary layer. The position of the stagnation point in a boundary layer is related to the start of its separation; in the core of the vortex, to the start of its decay. In the second case, one can draw an analogy with gasdynamics: the breakdown of the solution from the mathematical point of view takes place for identical reasons. A description of the space evolution of the vortex core has been given, and a flow pattern proposed in the region of its decay. A quantitative estimate of the height of a tornado-like vortex, as the point of its decay, has been obtained.

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